11.5 Nonlinear Systems of Different Equations

2. \[
\frac{dx_1}{dt} = 3x_1^2x_2 - 6x_1^2 \\
\frac{dx_2}{dt} = x_1x_2^2 - 2x_2^2
\]
Find the equilibrium point.

\[
\frac{dx_1}{dt} = 0 = 3x_1^2x_2 - 6x_1^2, \quad \frac{dx_2}{dt} = 0 = x_1x_2^2 - 2x_2^2 \\
x_2 = 2, \quad x_1 = 2
\]

<table>
<thead>
<tr>
<th>region</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(dx_1/dt)</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(dx_2/dt)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

4. \[
\frac{dx_1}{dt} = x_1x_2^3 - 2x_1^2x_2^2 \\
\frac{dx_2}{dt} = 9x_1^3x_2 - 3x_1^3x_2
\]
Find the equilibrium point.

\[
\frac{dx_1}{dt} = 0 = x_1x_2^3 - 2x_1^2x_2^2 \\
x_1x_2^3 = 2x_1^2x_2^2 \\
x_2 = 2,
\]

\[
\frac{dx_2}{dt} = 0 = 9x_1^3x_2 - 3x_1^3x_2 \\
9x_1^3x_2 = 3x_1^3x_2 \\
3 = x_1
\]

6. (a) \[
\frac{dx_1}{dt} = x_2^2 - x_1x_2 - x_2 \\
\frac{dx_2}{dt} = 2x_1^2 + x_1x_2 - 7x_1
\]
Find the equilibrium point.

\[
\frac{dx_1}{dt} = 0 = x_2^2 - x_1x_2 - x_2 \\
x_2 = x_1 + 1 \quad \text{eq.(1)},
\]

\[
\frac{dx_2}{dt} = 0 = 2x_1^2 + x_1x_2 - 7x_1 \\
x_2 = -2x_1 + 7 \quad \text{eq.(2)},
\]
Solving equations (1) and (2) simultaneously gives

\[x_1 = 2, x_2 = 3.\]

(b) Plot equations (1) and (2) and test each region.

8. (a) \[
\frac{dx_1}{dt} = x_1^2x_2 - x_1^3 \\
\frac{dx_2}{dt} = 6x_2^2 + x_3^3 - 3x_1x_2^2
\]
Find the equilibrium point.

\[
\frac{dx_1}{dt} = 0 = x_1^2x_2 - x_1^3 \\
x_2 = x_1 \quad \text{eq.(1)},
\]

\[
\frac{dx_2}{dt} = 0 = 6x_2^2 + x_3^3 - 3x_1x_2^2 \\
0 = 6 + x_2 - 3x_1 \\
x_2 = 3x_1 - 6 \quad \text{eq.(2)},
\]
Solving equations (1) and (2) simultaneously gives

\[ x_1 = 3, x_2 = 3. \]

(b) Plot equations (1) and (2) and test each region.

The figure shows that if both populations are greater than their equilibrium values they will increase without bound and if they are less than their equilibrium values they will decrease toward zero.

12. \( \frac{dy_1}{dt} = k_1 y_1 (1 - F_1 - y_1 - by_2) \)

\[ \frac{dy_2}{dt} = k_2 y_2 \left( 1 - F_2 - \frac{y_2}{y_1} \right) \]

Note that \( k_1, k_2, \) and \( b \) are greater than zero.

(a) \( F_1 < 1 \) and \( F_2 > 1 \)

\( \Rightarrow \frac{dy_1}{dt} < 0 \) so the whale population would decrease. With \( F_1 < 1, \frac{dy_1}{dt} \) can be either positive or negative, but it will always be greater than if \( F_1 \) were greater than 1. Thus, the krill population will increase more quickly or decrease more slowly, depending on the value of \( F_1 \).

(b) \( F_1 > 1 \) and \( F_2 > 1 \)

\( \Rightarrow \frac{dy_2}{dt} < 0 \) so the krill population will decrease. \( \frac{dy_2}{dt} < 0 \) so the whale population would decrease.

14. (a) Set the derivatives to zero.

\[ \frac{dx_1}{dt} = 0, = r_1 x_1 \left( 1 - \frac{x_1}{k_1} - b_1 \frac{x_2}{k_2} \right) \]

\[ \frac{dx_2}{dt} = 0, = r_2 x_2 \left( 1 - \frac{x_2}{k_2} - b_2 \frac{x_1}{k_1} \right) \]

Substitute \( k_1 = 6, k_2 = 4, b_1 = 2, \) and \( b_2 = 1. \)

\[ r_1 x_1 \left( 1 - \frac{x_1}{6} - \frac{2 x_2}{6} \right) = 0 \]

\[ r_2 x_2 \left( 1 - \frac{x_2}{4} - \frac{x_1}{4} \right) = 0 \]

The solution to this system is \( x_1 = 2, x_2 = 2. \)